Comment on Dirac spectral sum rules for QCD_3

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Recently Magnea [Phys. Rev. **D61**, 056005 (2000); Phys. Rev. **D62**, 016005 (2000)] claimed to have computed the first sum rules for Dirac operators in 3D gauge theories from 0D non-linear σ models. I point out that these computations are incorrect, and that they contradict with the exact results for the spectral densities unambiguously derived from random matrix theory by Nagao and myself.

Magnea [1] has recently claimed to have derived Dirac spectral sum rules for three-dimensional gauge theories coupled in a $(P, \mathbf{Z}_2)$-invariant manner to fundamental fermions with $N_c = 2$ (corresponding to the Dyson index $\beta = 1$) and adjoint fermions ($\beta = 4$). She employed the small-mass expansion of the low-energy effective theories, i.e. the zero-dimensional σ models over Riemannian symmetric spaces $\mathcal{M} =$ CII and BDI, instead of AIII that had been proposed for the case with fundamental fermions and $N_c \geq 3$ ($\beta = 2$) [2]. She concluded that the first sum rule in the presence of even number (N_f) of massless 2-component complex or 4-component real (Majorana) fermions is common both to the $\beta = 1$ and $\beta = 4$ universality classes, and takes the form†

$$\left\langle \sum_i \frac{1}{\zeta_i^2} \right\rangle^{(1,4)} = \int_{-\infty}^{\infty} d\zeta \frac{\rho_s^{(1,4)}(\zeta)}{\zeta^2} = \frac{4}{N_f}, \tag{1}$$

where ζ stands for an unfolded Dirac eigenvalue (i.e. rescaled by $1/(\pi\rho(0))$) and $\rho_s^{(\beta)}(\zeta)$ stands for the scaled spectral density. If true, this conclusion, derived from an obviously correct formula (see Ref. [3])

$$\left\langle \sum_i \frac{1}{\zeta_i^2} \right\rangle = \frac{d^2}{N_f\, M} \tag{2}$$

(d stands for the rank of the matrix that parameterizes $\mathcal{M}$, and M for the dimension of $\mathcal{M}$), would be surprising, as the four-dimensional counterpart of the spectral sum rules are known to be different for three values of β [4,3].

On the other hand, Nagao and myself [5] have obtained Pfaffian expressions for the generic p-level correlation functions in a presence of an arbitrary number of (for $\beta = 1$) and an arbitrary number of pairwise degenerate (for $\beta = 4$) finite fermion masses $\{\mu_f\}$, by applying the skew-orthogonal polynomial method to pertinent random matrix ensembles. To make comparison with Eq.(1), I take a completely confluent limit $\mu_f \to 0$ for all f, of our results [Ref. [5], Eqs.(2.40), (2.42), (3.23), (3.25), (3.49), (3.51)] with $p = 1$ (spectral density), to obtain:

$$\pi\, \rho_s^{(1)}(\zeta) = 1 - \frac{3}{\zeta^2} + \frac{3}{2\zeta^4} - \frac{3\cos 2\zeta}{2\zeta^4} \qquad (N_f = 2), \tag{3a}$$

$$= 1 - \frac{10}{\zeta^2} - \frac{30}{\zeta^4} + \frac{210}{\zeta^6} + \frac{525}{2\zeta^8} + \left(\frac{70}{\zeta^5} - \frac{525}{\zeta^7}\right)\sin 2\zeta + \left(-\frac{5}{\zeta^4} + \frac{315}{\zeta^6} - \frac{525}{2\zeta^8}\right)\cos 2\zeta \qquad (N_f = 4), \tag{3b}$$

$$\begin{aligned} &= 1 - \frac{21}{\zeta^2} - \frac{357}{2\zeta^4} - \frac{945}{\zeta^6} + \frac{48195}{\zeta^8} + \frac{218295}{\zeta^{10}} + \frac{1964655}{2\zeta^{12}} + \left(\frac{378}{\zeta^5} - \frac{50652}{\zeta^7} + \frac{873180}{\zeta^9} - \frac{1964655}{\zeta^{11}}\right)\sin 2\zeta \\ &\quad + \left(-\frac{21}{2\zeta^4} + \frac{5859}{\zeta^6} - \frac{266490}{\zeta^8} + \frac{1746360}{\zeta^{10}} - \frac{1964655}{2\zeta^{12}}\right)\cos 2\zeta \qquad (N_f = 6), \end{aligned} \tag{3c}$$

$$\pi\, \rho_s^{(4)}(\zeta) = 1 - \frac{\sin 2\zeta}{2\zeta} \qquad (N_f = 2), \tag{4a}$$

$$= 1 - \frac{\sin^2 2\zeta}{4\zeta^2} + \left(-\frac{\sin 2\zeta}{4\zeta^2} + \frac{\cos 2\zeta}{2\zeta}\right)\mathrm{Si}\,(2\zeta) \qquad (N_f = 4), \tag{4b}$$

$$= 1 - \frac{3}{4\zeta^2} + \frac{3}{32\zeta^4} + \left(\frac{1}{\zeta} - \frac{3}{4\zeta^3}\right)\sin 2\zeta + \frac{3\cos 2\zeta}{2\zeta^2} - \frac{3\cos 4\zeta}{32\zeta^4} \qquad (N_f = 6), \tag{4c}$$

$$\begin{aligned} &= 1 - \frac{27}{16\zeta^2} + \frac{45}{64\zeta^4} + \frac{45}{256\zeta^6} + \left(\frac{15}{32\zeta^3} - \frac{45}{64\zeta^5}\right)\sin 4\zeta + \left(-\frac{3}{16\zeta^2} + \frac{45}{64\zeta^4} - \frac{45}{256\zeta^6}\right)\cos 4\zeta \\ &\quad + \left[\left(\frac{9}{4\zeta^2} - \frac{45}{32\zeta^4}\right)\sin 2\zeta + \left(-\frac{3}{4\zeta} + \frac{45}{16\zeta^3}\right)\cos 2\zeta\right]\mathrm{Si}\,(2\zeta) \qquad (N_f = 8), \end{aligned} \tag{4d}$$

(Si stands for the sine-integral function) and so forth. These expressions for the spectral densities lead to the sum rules

$$\int_{-\infty}^{\infty} d\zeta \frac{\rho_s^{(1)}(\zeta)}{\zeta^2} = \frac{N_f}{(N_f-1)(2N_f+1)}, \tag{5}$$

$$\int_{-\infty}^{\infty} d\zeta \frac{\rho_s^{(4)}(\zeta)}{\zeta^2} = \frac{N_f}{(N_f-1)(N_f/2+1)}, \tag{6}$$

which are sensitive to the Dyson index β. The above sum rules agree perfectly with the numerical results for random matrix ensembles of large but finite ranks (~ 40), obtained by Hilmoine and Niclasen [6] via two alternative methods (an analytical method of Widom's [Table 4 of Ref. [6]] and numerical Monte-Carlo simulations of random matrix ensembles). Therefore I conclude that the expression (1) is erroneous, and the coincidence of the first sum rules for $\beta = 1$ and $\beta = 4$ claimed in her papers is illusory.

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† N_f in Ref. [1] stands for the number of 4-component spinors, i.e. a half of N_f in this note and in Refs. [2,5,6].

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